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AN ELLIPSOIDAL CRACK AND NEEDLE IN AN ANISOTROPIC ELASTIC MEDIUM

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We consider an ellipsoidal crack and needle (of small, but finite thickness) in an anisotropic elastic medium and a homogeneous external field. We obtain and investigate the explicit expressions for the stresses on their surfaces. We show that by decreasing the thickness of the needle the stresses corresponding to any load tend to a finite limit, i.e. they do not contain singularities, while for the ellipsoidal crack a singularity arises if the external field contains a component normal to the plane of the crack. In the case of extension the maximum stress is always attained at the edge of the crack while in the case of pure shear, as a rule, in its small neighborhood. In the latter case there is a sharp peak of stresses and on the edge itself all the components of the stress tensor may vanish. This points to the necessity of investigating the stresses on the entire surface of the crack and not only in its characteristic points.

By an ellipsoidal crack (needle) we will undestand an ellipsoidal cavity having one small (large) dimension in comparison with the other two dimensions. This allows us, at the computation of the stresses at the surface of the cavity, to restrict ourselves to the principal term of the expansion with respect to a corresponding small but finite parameter. The limiting case when the parameter tends to zero corresponds to the elliptic crack. In this case, only the stresses in the neighborhood of the crack or the limiting values of the nonsingular components of the stress tensor on the crack itself are meaningful.

In most cases (see, for example [1] where there are other references) the elliptic cracks have been studied. The results for an ellipsoidal crack in an isotropic medium can be obtained by a limiting process from the known solutions for the ellipsoidal cavity constructed in [2 – 4] and which has been done in [5, 6]. However, the components of the stress tensor have been studied not on the entire surface of the crack but only at its edge. The stresses at the vertices of a spheroidal crack and needle in a transversely isotropic medium have been obtained in [7] for external fields which do not have singularities.

As opposed to the mentioned papers, here we consider an arbitrary anisotropic medium and we investigate the complete state of stress on the entire surface of an ellipsoidal crack and needle. Such an investigation turns out to be essential since in some cases there is an abrupt increase of stresses near the edge of the crack, although on the edge itself they are equal to zero.

In this paper we make use of the general solution of the stress concentration problem on the surface of an ellipsoidal cavity in an anisotropic medium which has been obtained in [8]. In Sect. 1 we give some formulas from [8] which are necessary in what follows and we introduce parameters which are convenient for the limitimg processes. In Sects. 2 and 3 we obtain expressions for the stresses on the surface of the ellipsoidal needle and crack. In Sects 4 and 5 we give a complete study of the stress concentrations at the surface of an ellipsoidal crack in an isotropic and also in an orthotropic medium for all cases of external homogeneous fields.

1. The stresses $\sigma^{\alpha\beta}$ (n) at the surface of an ellipsoidal cavity in a homogeneous external field $\sigma_0{}^{\lambda\mu}$ have the form [8]

$$\sigma^{\alpha\beta}(\mathbf{n}) = F^{\alpha\beta}{}_{\cdot\cdot\lambda\mu}(\mathbf{n})\,\sigma_0^{\lambda\mu} \tag{1.1}$$

Here F (n) is the tensor coefficient of the stress concentration depending on the normal n to the surface of the ellipsoid

$$\mathbf{x}a^{-2}\mathbf{x} = x^\alpha (a^{-2})_{\alpha\beta} x^\beta = 1, \qquad a^{\alpha\beta} = a^\alpha \delta^{\alpha\beta} \tag{1.2}$$

with semiaxes a^α ($\alpha = 1, 2, 3$). The relation between the coordinates x^α of the points of the ellipsoid and the normal n_β is given by

$$\mathbf{x} = \frac{a^2\mathbf{n}}{\sqrt{\mathbf{n}a^2\mathbf{n}}}, \qquad \mathbf{n} = \frac{a^{-2}\mathbf{x}}{\sqrt{\mathbf{x}a^{-4}\mathbf{x}}} \tag{1.3}$$

We represent the coefficient F (n) of the concentration in the form of the product of two factors

$$F^{\alpha\beta}{}_{\cdot\cdot\lambda\mu}(\mathbf{n}) = B^{\alpha\beta\sigma\tau}(\mathbf{n})\,(B_0^{-1})_{\sigma\tau\lambda\mu} \tag{1.4}$$

the first of which B (n) depends explicitly only on the tensor of the elastic constants c_0 of the medium, while it depends implicitly on the parameters a^α of the ellipsoid through the normal n. In the limiting cases of a needle and a crack B (n) does not vary and, consequently, does not have singularities. In particular, for the isotropic medium with shear modulus μ_0 and Poisson's ratio ν_0 ($\varkappa_0 = 2\mu_0 / 1 - \nu_0$)

$$B^{\alpha\beta\sigma\tau}(\mathbf{n}) = \varkappa_0\Big[\nu_0(\delta^{\alpha\beta}\delta^{\sigma\tau} - n^\alpha n^\beta \delta^{\sigma\tau} - n^\sigma n^\tau \delta^{\alpha\beta}) + \frac{1-\nu_0}{2}(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma} - n^\alpha n^\sigma \delta^{\beta\tau} - n^\alpha n^\tau \delta^{\beta\sigma} - n^\beta n^\sigma \delta^{\alpha\tau} - n^\beta n^\tau \delta^{\alpha\sigma}) + n^\alpha n^\beta n^\sigma n^\tau\Big] \tag{1.5}$$

where $\delta^{\alpha\beta}$ is the Kronecker symbol. The second factor in (1.4) is a constant tensor inverse of the tensor

$$B_0^{\sigma\tau\lambda\mu} = \frac{|\det a|}{4\pi}\int B^{\sigma\tau\lambda\mu}(\mathbf{n})\frac{d\mathbf{n}}{(\mathbf{n}a^2\mathbf{n})^{3/2}} \tag{1.6}$$

depends on the parameters of the ellipsoid (1.2) and, as it will be shown, becomes singular in the case of a crack.

For the limiting processes it is convenient to introduce the dimensionless parameters

$$\alpha = \frac{a_2}{a_1}, \qquad \varepsilon = \frac{a_3}{a_1}, \qquad \xi = \frac{a_3}{a_2} \qquad (a_1 \geqslant a_2 \geqslant a_1) \tag{1.7}$$

The case $\alpha \ll 1$, $\xi \sim 1$ corresponds to the needle, while $\xi \ll 1$, $\alpha \sim 1$ to the crack, and $\varepsilon \ll \alpha \ll 1$ to a narrow crack. We note that in all cases we have $\varepsilon = \xi\alpha \ll 1$.

Thus, the solution of the stress concentration problems for a needle and a crack reduces to the computation of the principal terms in the expansion of the tensor $B_0{}^{-1}$ with respect to the corresponding small parameter. First we carry out this computation for a needle.

2. In (1.6) we switch to the spherical coordinates φ, θ with the polar axis along the axis of the needle, i.e. along x^1. We perform the change of variable $\cos\theta = t$ and we set

$$\alpha_1 = \alpha \sqrt{\cos^2\varphi + \xi^2 \sin^2\varphi}, \qquad B(\varphi, t) = B(\mathbf{n}(\varphi, t)) \tag{2.1}$$

Without loss of generality we can assume that $B(\varphi, t)$ is an even function of t (only this component gives a contribution in the integral (1.6)).

We have

$$B_0 = \frac{\xi}{2\pi} \int_0^{2\pi} \frac{d\varphi}{\cos^2\varphi + \xi^2 \sin^2\varphi} \int_{-1}^{1} B(\varphi, t) f_1(t, \alpha_1)\, dt$$

$$f_1(t, \alpha_1) = \frac{\alpha_1^2}{2\,[\alpha_1^2 + (1-\alpha_1^2)\,t^2]^{3/2}}$$

For a needle $\alpha \to 0$, $\xi \sim 1$ and $f_1(t, \alpha_1) \to \delta(t)$. Consequently, the principal term in the expansion of B_0 with respect to α has the form

$$B_{00} = \frac{\xi}{2\pi} \int_0^{2\pi} \frac{B(\varphi, 0)}{\cos^2\varphi + \xi^2 \sin^2\varphi}\, d\varphi \tag{2.2}$$

Thus, for an arbitrary anisotropic medium the problem is reduced to the computation of a simple integral. If the tensor B_{00} has an inverse ($\det B_{00} \neq 0$), then the coefficient of the concentration tends to a constant value when $\alpha \to 0$, i.e. it has no singularity. The computations show that $\det B_{00} \neq 0$ if the symmetry of the medium is not below the rhombic symmetry (orthotropic, hexagonal, cubic and others (*)). Obviously, this holds also in the case of an arbitrary anisotropy. The tensor of the elastic constants of the indicated media has nine nonzero components, denoted according to the usual rule by

$$c_0^{\alpha\alpha\beta\beta} = c_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, 3)$$

$$c_0^{2323} = c_{44}, \quad c_0^{1313} = c_{55}, \quad c_0^{1212} = c_{66}$$

For the orthotropic medium all the nine components are essential, for the transversely isotropic medium only five components are essential

$$c_{11} = c_{22},\ c_{12},\ c_{13} = c_{23},\ c_{33},\ c_{44} = c_{55},\ c_{66} = {}^1/_2\,(c_{11} - c_{12})$$

for the cubic symmetry we have three constants

$$c_{11} = c_{22} = c_{33},\ c_{12} = c_{13} = c_{23},\ c_{44} = c_{55} = c_{66}$$

and, finally, for the isotropic medium

$$c_{12} = \lambda_0,\ c_{44} = {}^1/_2\,(c_{11} - c_{12}) = \mu_0,\ c_{11} = \lambda_0 + 2\mu_0$$

In all these cases the tensor B_{00}^{-1} is found in explicit form. For its computation it is necessary to take $n_1 = 0$ in the tensor $B(\mathbf{n})$ of [8], insert the obtained expression into (2.2), integrate and invert the tensor B_{00}. However for the component of B_{00}^{-1} with indices $(\alpha\alpha\beta\beta)$ corresponding to extension along the axes, the expressions become cumbersome because it is necessary to invert a matrix of order three. Therefore here we give only the nonzero shear components

$$(B_{00}^{-1})_{1212} = \frac{\sqrt{c_{55}} + \xi\sqrt{c_{66}}}{c_{66}\sqrt{c_{55}}}, \qquad (B_{00}^{-1})_{1313} = \frac{\sqrt{c_{55}} + \xi\sqrt{c_{66}}}{\xi c_{55}\sqrt{c_{66}}}$$

*) Here and in the following we assume that the symmetry axes of the medium coincide with the axes of the ellipsoid.

$$(B_{00}^{-1})_{2323} = \frac{c_{33}}{c_{22}c_{33} - c_{23}^2} \frac{1}{\xi(\xi A_1 \sqrt{u_1 u_2} + B_1 \sqrt{|u_2|} + C_1 \sqrt{|u_1|})} \tag{2.3}$$
$$A_1 = -\frac{1}{(1+u_1\xi^2)(1+u_2\xi^2)}, \quad B_1 = -\frac{u_1}{(1+u_1\xi^2)(u_2-u_1)}$$
$$C_1 = -\frac{u_2}{(1+u_2\xi^2)(u_1-u_2)}$$

Here u_1 and u_2 are the roots of the quadratic equation

$$c_{33}c_{44}u^2 + (c_{22}c_{33} - 2c_{23}c_{44} - c_{23}^2)\,u + c_{22}c_{44} = 0$$

The computations are considerably simplified in the isotropic case. Taking into account (1.5), we obtain the nonzero components of B_{00}^{-1} ($\eta_0 = [2\mu_0(1+\nu_0)]^{-1}$)

$$(B_{00}^{-1})_{1111} = \eta_0, \qquad (B_{00}^{-1})_{1122} = (B_{00}^{-1})_{1133} = -\nu_0\eta_0$$
$$(B_{00}^{-1})_{2233} = -\eta_0(1-2\nu_0^2), \qquad (B_{00}^{-1})_{2222} = \eta_0[1+2(1-\nu_0^2)\xi]$$
$$(B_{00}^{-1})_{3333} = \eta_0\left[1 + \frac{2(1-\nu^2)}{\xi}\right], \qquad (B_{00}^{-1})_{1212} = \frac{1+\xi}{4\mu_0} \tag{2.4}$$
$$(B_{00}^{-1})_{1313} = \frac{1}{4\mu_0}\left(1+\frac{1}{\xi}\right), \qquad (B_{00}^{-1})_{2323} = \frac{1}{2\varkappa_0}\frac{(1+\xi)^2}{\xi}$$

Inserting B_{00}^{-1} into (1.4) we can obtain, according to (1.1), the stresses $\sigma(\mathbf{n})$ at the surface of the needle for an arbitrary external field σ_0. We note that the values of $\sigma(\mathbf{n})$ coincide with those obtained in [7] at the characteristic points of the needle.

The expressions (2.3) and (2.4) give an obvious mechanism for the appearance of the singularity at the conversion from a needle to a narrow crack, i.e. for $\xi \to 0$: a singularity of order ξ^{-1} appears only for external stresses σ_0 containing components with the index 3.

3. We consider an arbitrary crack $\xi \ll 1$, $\alpha \sim 1$. In the same way as in the case of a needle, we switch in (1.6) to spherical coordinates φ, θ, but with the polar axis oriented along x^3. Setting

$$\xi_1 = \frac{\xi\alpha}{\sqrt{\cos^2\varphi + \alpha^2\sin^2\varphi}}, \qquad t = \cos\theta \tag{3.1}$$

and maintaining the definition (2.1) for $B(\varphi, t)$, we find

$$B_0 = \frac{\alpha}{2\pi}\int_0^{2\pi} \frac{d\varphi}{\cos^2\varphi + \alpha^2\sin^2\varphi} \int_{-1}^{1} B(\varphi, t) f_2(t, \xi_1)\, dt$$
$$f_2(t, \xi_1) = \frac{\xi_1}{2[1-(1-\alpha^2\xi_1^2)t^2]^{3/2}} \tag{3.2}$$

We write the expansion of B_0 as a function of ξ in the form

$$B_0 = B_{00} + \xi B_{01} + O(\xi^2) \tag{3.3}$$

As opposed to the case of the needle, for the computation of the principal term in the expansion of B_0^{-1} with respect to ξ we have to retain the first two terms in B_0 since the tensor B_{00} in (3.3) does not have an inverse, i.e. $\det B_{00} = 0$. This can be verified by simple computations if the symmetry of the medium is not below the rhombic symmetry. In order to show that $\det B_{00} = 0$, it is sufficient to show that all the components of the tensor B_{00} containing the index 3 are equal to zero. In fact, as it will be shown below, in the integral for B_{00} there occurs the tensor B $(\varphi, 1)$ which corresponds to B $(\mathbf{n})$ for $n_1 = n_2 = 0$, $n_3 = 1$. But from the general expression for $B(\mathbf{n})$, given in [8], it follows that in the case of a rhombic structure, the corresponding components of B $(\mathbf{n})$ which contain the index 3 are equal to zero for $n_1 = n_2 = 0$. We should expect that this is true also for an arbitrary anisotropy.

For the computation of B_{00} and B_{01}, we will consider $f_2(t, \xi_1)$ as a generalized function on the segment $|t| \leqslant 1$ with identified points ± 1, i.e. on the circumference. This is possible by virtue of the fact that B (φ, t) is even and continuous. We can verify that for $\xi_1 \to 0$

$$f_2(t, \xi_1) = \delta(t \pm 1) + \frac{\xi_1}{2(1-t^2)^{3/2}} + O(\xi_1^2)$$

Substitution into (3.2) gives

$$B_{00} = \frac{\alpha}{2\pi} \int_0^{2\pi} \frac{B(\varphi, \pm 1)}{\cos^2\varphi + \alpha^2 \sin^2\varphi} d\varphi \tag{3.4}$$

$$B_{01} = \frac{\alpha^2}{4\pi} \int_0^{2\pi} \frac{d\varphi}{(\cos^2\varphi + \alpha^2\sin^2\varphi)^{3/2}} \int_{-1}^{1} \frac{B(\varphi, t) - B(\varphi, \pm 1)}{(1-t^2)^{3/2}} dt$$

where the latter integral is written in regularized form. We note that the obtained expressions become considerably simpler for the circular $(\alpha = 1)$ and for the narrow $(\alpha \ll 1)$ crack. In the isotropic case the integrals (3.4) can be easily computed and the inversion of the tensor (3.3) allows us to obtain explicit expressions for the components of the principal term of the expansion of B_0^{-1}

$$(B_0^{-1})_{1313} = \frac{1}{2\varkappa_0} \frac{1-\alpha^2}{(1-\nu_0-\alpha^2)E(\sqrt{1-\alpha^2}) + \alpha^2\nu_0 K(\sqrt{1-\alpha^2})} \frac{1}{\xi}$$

$$(B_0^{-1})_{2323} = \frac{1}{2\varkappa_0} \frac{1-\alpha^2}{(1-\alpha^2-\nu_0\alpha^2)E(\sqrt{1-\alpha^2}) - \nu_0\alpha^2 K(\sqrt{1-\alpha^2})} \frac{1}{\xi}$$

$$(B_0^{-1})_{3333} = \frac{2}{\varkappa_0} \frac{1}{E(\sqrt{1-\alpha^2})} \frac{1}{\xi} \tag{3.5}$$

Here K (α) and E (α) are complete elliptic integrals of the first and the second kind, respectively. The remaining components can be considered equal to zero with the accuracy of O (1). From here it follows that a contribution in the singular stresses $\sigma(\mathbf{n})$ at the surface of the crack is given only by the components of the external field σ_0 which have index equal to 3. This is in agreement with the results in [5, 6].

Passing to the narrow crack $\alpha \to 0$ and

$$(B_0^{-1})_{1313} = \frac{1}{4\mu_0}, \qquad (B_0^{-1})_{2323} = \frac{1}{2\varkappa_0} \frac{1}{\xi} \tag{3.6}$$

$$(B_0^{-1})_{3333} = \frac{2}{\varkappa_0} \frac{1}{\xi}$$

which coincides with (2.4) for $\xi \to 0$. In the case when the medium has a rhombic symmetry, the tensor B_0^{-1} has a similar structure and its components can be expressed in the general case in terms of elliptic integrals. They become considerably simpler for the narrow crack and have the form

$$(B_0^{-1})_{1313} = \frac{1}{4\sqrt{c_{55}c_{66}}} \frac{1}{\xi}, \qquad (B_0^{-1})_{2323} = \frac{c_{33}}{4(c_{22}c_{33} - c_{23}^2)} \frac{\sqrt{|u_1|} + \sqrt{|u_2|}}{\xi}$$

$$(B_0^{-1})_{3333} = \frac{\sqrt{c_{22}c_{33}}}{c_{22}c_{33} - c_{23}^2} \frac{\sqrt{|u_1|} + \sqrt{|u_2|}}{\xi} \tag{3.7}$$

which, obviously, in the particular case of the isotropic medium coincide with (3.6).

We emphasize that, representing an independent interest, the case of the narrow crack not only makes obvious the mechanism of the variation of the stress concentration when passing from a needle to a crack, but also allows us to effect the transfer to the plane problem if we consider the median sector of the crack.

4. We consider now the direct investigation of the stresses at the surface of the crack. We start with pure extension. Since the components σ_0^{11} and σ_0^{22} according to (3.5)–(3.7) do not give a contribution to the singularity, we consider only the extension σ_0^{33} along the axis x^3 normal to the crack.

From (1.1), (1.4) and (3.5) – (3.7) we find

$$\sigma^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta 33}(\mathbf{n})(B_0^{-1})_{3333}\,\sigma_0^{33} \tag{4.1}$$

It is convenient to carry out the investigation of $\sigma^{\alpha\beta}(\mathbf{n})$ in a local system of coordinates $x^{\alpha'}$, connected at each point of the surface of the ellipsoidal crack with the normal $\mathbf{n}$. As a local basis we take

$$\mathbf{e}_{3'} = \mathbf{n}, \quad \mathbf{e}_{1'} = \mathbf{n}^\circ \times \mathbf{e}_3, \quad \mathbf{e}_{2'} = \mathbf{n} \times (\mathbf{n}^\circ \times \mathbf{e}_3)$$

$$\mathbf{n}_0 = \frac{1}{\sqrt{n_1^2 + n_2^2}}(n_1\mathbf{e}^1 + n_2\mathbf{e}^2)$$

Here $\mathbf{e}_\alpha$ are the unit vectors of the coordinate system connected with the axes of the ellipsoid, while the vector $\mathbf{n}^\circ$ is normal to the border of the crack at the point $(n_1, n_2, 0)$. It is obvious that for the points of the surface, which represent a fundamental interest, near the edge, the axes $x^{1'}$ and $x^{2'}$ are respectively parallel and normal to the edge. With respect to the local axes, the tensor $\sigma^{\alpha'\beta'}(\mathbf{n})$ is plane since all components with index 3 are equal to zero. This follows from the equilibrium conditions.

First we consider the isotropic case. Switching in (4.1) to local coordinates and taking into account (1.5), (3.5) and (3.6), we find

$$\sigma^{1'1'}(\mathbf{n}) = \nu_0\sigma_{\max}(1 - n_3^2), \quad \sigma^{2'2'}(\mathbf{n}) = \sigma_{\max}(1 - n_3^2), \quad \sigma^{1'2'}(\mathbf{n}) = 0 \tag{4.2}$$

Here for the finite and the narrow crack we have, respectively,

$$\sigma_{\max} = \frac{2}{E(\sqrt{1-\alpha^2})}\,\frac{1}{\xi}\,\sigma_0^{33}, \qquad \sigma_{\max} = \frac{2}{\xi}\,\sigma_0^{33} \tag{4.3}$$

Thus, for the given load the selected system of coordinates coincides, with the accepted degree of accuracy, with the principal axes of the tensor $\sigma(\mathbf{n})$. Since the principal stresses $\sigma_1 = \sigma^{1'1'}$ and $\sigma_2 = \sigma^{2'2'}$ have the same sign, the maximal tangential stress is $\tau_{\max} = {}^1/_2\,|\,\sigma_2\,|$. It is clear from (4.2) that in the case of extension the largest value of the stresses in any cross section normal to the edge is attained for $n_3 = 0$, i.e. on the edge itself.

In the anisotropic case the computations are similar but more cumbersome. We give the final expressions for the stresses $\sigma(\mathbf{n})$ at the surface of a narrow crack in an orthotropic medium in the cross sections $n_1 = 0$ and $n_2 = 0$. In the selected local basis which in the present case coincides with the principal axes of the tensor $\sigma(\mathbf{n})$, for the cross section $n_2 = 0$ we have

$$\sigma_1(\mathbf{n}) = \Delta^{-1} c_{55}\,[c_{11}c_{23} - c_{12}c_{13})n_1^2 + (c_{12}\,c_{33} - c_{13}\,c_{23})\,n_3^2]n_1^2\,(B_0^{-1})_{3333}\,\sigma_0^{33} \tag{4.4}$$

$$\sigma_2(\mathbf{n}) = \Delta^{-1} c_{55}(c_{11}c_{33} - c_{13}^2)n_1^2 (B_0^{-1})_{3333}\,\sigma_0^{33}$$
$$\Delta = c_{11}c_{55}n_1^4 + (c_{11}c_{33} - 2c_{13}c_{55} - c_{13}^2)\,n_1^2 n_3^2 + c_{33}c_{55}n_3^4 \qquad (4.5)$$

Here $(B_0^{-1})_{3333}$ is given by the expression (3.7). The stresses in the cross section $n_1 = 0$ are obtained by replacing in the right-hand sides the indices 1 and 5 by 2 and 4, respectively.

5. We consider now pure shear. The component σ_0^{12} does not give a contribution to the singularity and the situation for σ_0^{13} and σ_0^{23} is similar. Therefore we assume that the external field coincides with σ_0^{13}.

In the isotropic case from (1.1), (1.4) and (3.5) we find

$$\sigma^{\alpha\beta}(\mathbf{n}) = 4B^{\alpha\beta 13}(\mathbf{n})\,(B_0^{-1})_{1313}\,\sigma_0^{13} \qquad (5.1)$$

If in the above introduced local system of coordinates we perform a rotation around the normal $\mathbf{n}$, i.e. the axis $x^{3'}$ through an angle

$$\varphi_0 = -\frac{1}{2}\operatorname{arctg}\frac{n_2}{n_1 n_3}$$

then the new axes are principal for $\sigma^{\alpha'\beta'}(\mathbf{n})$. For the principal stresses we find

$$\sigma_1(\mathbf{n}) = \sigma^{1'1'}(\mathbf{n}) = \sigma_{\max}\{(1-\nu_0)\sqrt{(1-n_1^2)(1-n_3^2)} - (1+\nu_0)n_1 n_3\} \qquad (5.2)$$
$$\sigma_2(\mathbf{n}) = \sigma^{2'2'}(\mathbf{n}) = -\sigma_{\max}\{(1-\nu_0)\sqrt{(1-n_1^2)(1-n_3^2)} + (1+\nu_0)n_1 n_3\}$$

Here for the finite and the narrow crack we have, respectively,

$$\sigma_{\max} = \frac{1-\alpha^2}{(1-\nu_0-\alpha^2)E(\sqrt{1-\alpha^2}) + \nu_0\alpha^2 K(\sqrt{1-\alpha^2})}\,\frac{1}{\xi}\,\sigma_0^{13}$$
$$\sigma_{\max} = \frac{1}{1-\nu_0}\,\frac{1}{\xi}\,\sigma_0^{13} \qquad (5.3)$$

The largest absolute value of the principal stresses σ_1 and σ_2, equal to $\sigma_{\max}$, is attained at the cross section $n_2 = 0$ at the points

$$n^* = \left(\pm\frac{1}{\sqrt{2}},\, 0,\, \mp\frac{1}{\sqrt{2}}\right), \qquad n^* = \left(\pm\frac{1}{\sqrt{2}},\, 0,\, \pm\frac{1}{\sqrt{2}}\right)$$

which are the points of maximum (minimum) of the functions $\sigma_1(\mathbf{n})$ and $\sigma_2(\mathbf{n})$. In this cross section σ_1 and σ_2 are equal to zero at the edge. It is essential that the point of maximum is situated at a very small distance from the edge of the crack, which according to (1.3) and (5.2) is

$$\Delta x^1 = \frac{a_1}{2}\varepsilon^2 + O(\varepsilon^4)$$

We can also show that the characteristic width of the peak is of order ε. By varying the cross section from $n_2 = 0$ to $n_1 = 0$ the magnitude of the peak decreases while the point of maximum approaches the edge coinciding with it for $n_1 = 0$. For the cross sections in which σ_1 and σ_2 are of the same sign, the character of the variation of $\tau_{\max}$ is evident. In particular, in the cross section $n_2 = 0$ the quantity $\tau_{\max}$ has a sharp peak of height $\sigma_{\max}/2$ at the point n^* and vanishes at the edge. In the case when σ_1 and σ_2 have different signs

$$\tau_{\max} = (1-\nu_0)\,\sigma_{\max}\sqrt{(1-n_1^2)(1-n_3^2)}$$

This quantity attains its maximum value equal to $(1-\nu_0)\,\sigma_{\max}$, at the edge in the cross section $n_1 = 0$.

Thus, in the case of shear, a curious increase of stresses takes place, which, if not taken into account, will lead by numerical computations to qualitatively irregular results on the stress concentration in the crack. This shows the necessity of investigating the stresses not only in the characteristic points of the contour of the crack, but also on its entire surface.

A similar effect takes place in an anisotropic medium. The principal stresses at the surface of a narrow crack in an orthotropic medium in the cross section $n_2 = 0$ are

$$\sigma_1(\mathbf{n}) = -4\Delta^{-1} c_{55} [(c_{11}c_{23} - c_{12}c_{13}) n_1^2 + (c_{12}c_{33} - c_{13}c_{23}) n_3^2] n_1 n_3 (B_0^{-1})_{1313} \sigma_0^{13}$$

$$\sigma_2(\mathbf{n}) = -4\Delta^{-1} c_{55} (c_{11}c_{33} - c_{13}^2) n_1 n_3 (B_0^{-1})_{1313} \sigma_0^{13}$$

Here Δ and $(B_0^{-1})_{1313}$ are given by the expressions (4.5) and (3.7), respectively.

In the cross section $n_1 = 0$

$$\sigma_1(\mathbf{n}) = -\sigma_2(\mathbf{n}) = 4 \frac{c_{55}c_{66}n_2}{c_{66}n_2^2 + c_{55}n_3^2} (B_0^{-1})_{1313} \sigma_0^{13}$$

Obviously, as in the isotropic case, the increase effect takes place in the cross section $n_2 = 0$.

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